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DELAYED FRACTURE IN VISCOELASTIC–PLASTIC SOLIDS†

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Abstract—In some hard, viscoelastic polymers crack growth is associated with the formation of a wedge of crazed material at the tip of the crack. The crazed material is formed from the bulk material by growth of holes at a given stress which can be likened to a yield stress for metals.

This paper deals with the time at which a penny-shaped crack surrounded by a wedge shaped ring of crazed material begins to propagate. The bulk material is considered viscoelastic and the crazed material in the wedge is modelled by the visco-plasticity model of Crochet. It is found that time-dependent plasticity in the wedge shortens the time to failure in comparison with time-independent plasticity.

INTRODUCTION

ONE "objection" to the use of glass-like polymers as structural materials is their apparently inconsistent behavior with respect to failure. Conditions under which fracture may be induced can vary widely and, to the casual observer, in an erratic way. Probably the most disconcerting factor is the ability of polymers to carry loads for some time only, the time depending on the magnitude of the load.

In order to better understand the load carrying ability of such viscoelastic materials, it is necessary to study the growth of cracks in these materials. The prime difficulty in pursuing such studies from the continuum mechanics viewpoint, is the fact that many hard polymers exhibit not only viscoelastic properties but also rate or time sensitive phenomena reminiscent of metal yield. Such phenomena may be associated either with microstructural decomposition of the material or with geometric changes due to necking.

There are repeated examples of crack growth in a variety of hard polymers where material deformation similar to metal yield at the tip of the crack is contained in a wedgelike domain. This phenomenon is observed in thin polymer sheets [1] like in metal sheets [2] or in heavy sections under conditions of plane strain [3, 4] and to some extent in Berry's experiments [5]. For the latter case it has been shown by Kambour [4] that the yielded material at the tip of the crack is "crazed" and of lower density than the bulk polymer. The lower density is achieved by void growth which commences, for a given stress or strain history, at any apparently well defined stress level [3, 6].

The net effect of the crazed material at the tip of the crack—under conditions of plane strain—is thus to provide a small domain along the crack axis over which the normal tensile stress remains more or less constant. To the extent that the crazed material also possesses time-dependent properties, the stress level within this domain varies also with

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time. One objective of this paper is, therefore, to study the effect of the time-dependence of the crazed material on the time of initial crack propagation.

We distinguish here between initial crack propagation and, what is often referred to as catastrophic crack growth. A crack may remain dormant for some time [7] before any crack propagation occurs. Provided the loading is constant or increases, the crack will then grow, slowly at first [1, 8], but with monotonically increasing speed until an apparent crack speed transition occurs to crack velocities on the order of about one tenth of the Rayleigh surface wave speed. This apparent transition may occur very rapidly and can therefore be mistaken as an instability although the actual instant of instability has passed much earlier. Work on the velocity transition [8] and this present paper points out that the initial moment of crack growth, after which instant the crack accelerates monotonically, may occur several orders of magnitudes in time before the rapid crack growth. In dealing only with the initial transition of a crack from its stationary to its motion stage we approach the problem of fracture in a very conservative manner in the sense that actual structure failure might occur very much later than our prediction might lead us to believe.

In this paper we shall consider the growth of a penny-shaped crack in a viscoplastic material. Although the linear theory of viscoelasticity is well understood, there is very little quantitative knowledge regarding non-linear viscoelasticity or viscoplasticity. However, since we are interested primarily in investigating the effect of viscoplasticity rather than be bound to precise, quantitative predictions, we may be so liberal as to accept the viscoplasticity model of Crochet [9] which contains most of the qualitative features of what one would expect of a more complete constitutive formulation. The Crochet model attempts to generalize the elastic–plastic stress–strain law by replacing the elastic portion by a linearly viscoelastic one and makes the yield stress dependent on the rate of deformation during the initial, linearly viscoelastic deformation phase. It turns out that even with this relatively simple material representation the mathematics of the problem become very complicated, and a more detailed material representation would most likely lead to mathematical intractability. The effect of temperature may be incorporated through time–temperature reduction if the assumption of thermo-rheological simplicity is justified [10].

MATERIAL REPRESENTATION AND FAILURE CRITERION

We have stated that the bulk material is to be represented by a linearly viscoelastic solid. The stress–strain equations for such a body are given, under isothermal conditions, by

$$
\begin{aligned}
s_{ij} &= \int_{-\infty}^{t} G_1(t-\tau)\frac{\partial e_{ij}(\tau)}{\partial \tau}\,\mathrm{d}\tau \\
s &= \int_{-\infty}^{t} G_2(t-\tau)\frac{\partial e(\tau)}{\partial \tau}\,\mathrm{d}\tau
\end{aligned}
\tag{1}
$$

where $G_1(t)$ and $G_2(t)$ are the relaxation moduli in shear and isotropic compression respectively, s_{ij} and e_{ij} denote the deviatoric parts of stress and strain tensors, while $\delta_{ij}s$ and $\delta_{ij}e$ are the hydrostatic parts of these tensors.

For materials exhibiting rate or load history sensitive plasticity Crochet [9] suggested a viscoelastic–plastic constitutive relation wherein the yield modulus Y depends on the history of loading; its value is given by

$$Y(t) = A + B\exp(-C\chi) \tag{2}$$

where A, B, C are material constants, and χ is a function, of the strain state

$$\chi = \{(\varepsilon_{ij}^{v} - \varepsilon_{ij}^{e})(\varepsilon_{ij}^{v} - e_{ij}^{e})\}^{\frac{1}{2}} \tag{3}$$

summation being implied by repeated indices; the superscripts v and e denote the viscoelastic and purely (short-time) elastic components of the strain. For strains increasing with time equation (2) asserts that faster loading corresponds to a higher yield stress, while under constant stress it implies that yield occurs at a time which is longer the lower the stress. For initially elastic response under rapid loading $\varepsilon_{ij}^{v} = \varepsilon_{ij}^{e}$ and $Y(0) = A+B$ while the minimum yield value is given by $Y(\infty) \doteq A$, provided $\varepsilon_{ij}^{v} - \varepsilon_{ij}^{e}$ is sufficiently large as may be the case for viscoelastic non-linear† polymers.

Next we need to consider the criterion of incipient fracture. We shall define fracture to start when the strain at the tip reaches a critical value [11–13]. This condition, known alternately as the critical crack opening or displacement criterion, is a sufficient criterion for fracture initiation, although, as pointed out earlier, it is not a sufficient criterion for catastrophic failure in viscoelastic materials. It has been used for metals by Goodier and Field [14] and Olesiak and Wnuk [15] and for simple viscoelastic materials by Williams [16].

THE STRESS AND STRAIN DISTRIBUTION AROUND THE CRACK

Consider the axisymmetric geometry in Fig. 1. The crack proper extends over the domain $0 \leq r \leq l$ while the viscoplastic material is contained in a wedge in the ring $l \leq r \leq a(t)$. Our immediate aim is to determine the displacement w normal to the crack plane at the crack end $r = l$.

The problem of a growing crack in a viscoelastic medium or that of a crack of constant length but subjected to a time variation in loading cannot, in general, be treated by the correspondence principle. For one important case, however, when the loading increases monotonically with time, Graham [17] has shown that the distribution of stresses and strains around a crack can be found by an extended correspondence principle. This restriction amounts to the requirement that in the corresponding elastic solution the stresses do not depend on the modulus of the material. It is further assumed in this analysis that that part of the crack surface over which the loading is applied increases monotonically with time, that is to say, the "plastic" wedge may remain constant or only increase with time. For a more complete discussion on the solution the reader is referred to Graham's papers [17].

The result for the normal displacement w in the crack plane $z = 0$, [17]

$$\begin{aligned} w(\rho, t) = {} & \frac{2}{\pi} K(o) \int_{\rho}^{a(t)} \frac{\mathrm{d}v}{(v^2-\rho^2)^{\frac{1}{2}}} \int_0^v \frac{sp(s,t)\,\mathrm{d}s}{(v^2-s^2)^{\frac{1}{2}}} \\ & + \frac{2}{\pi} \int_0^t K(\tau)\,\mathrm{Re}\left\{ \int_{\rho}^{a(t-\tau)} \frac{\mathrm{d}v}{(v^2-\rho^2)^{\frac{1}{2}}} \int_0^v \frac{sp(s,t-\tau)\,\mathrm{d}s}{(v^2-s^2)^{\frac{1}{2}}} \right\} \mathrm{d}\tau \end{aligned} \tag{4}$$

† Non-linear in the chemical sense of "un-crosslinked".

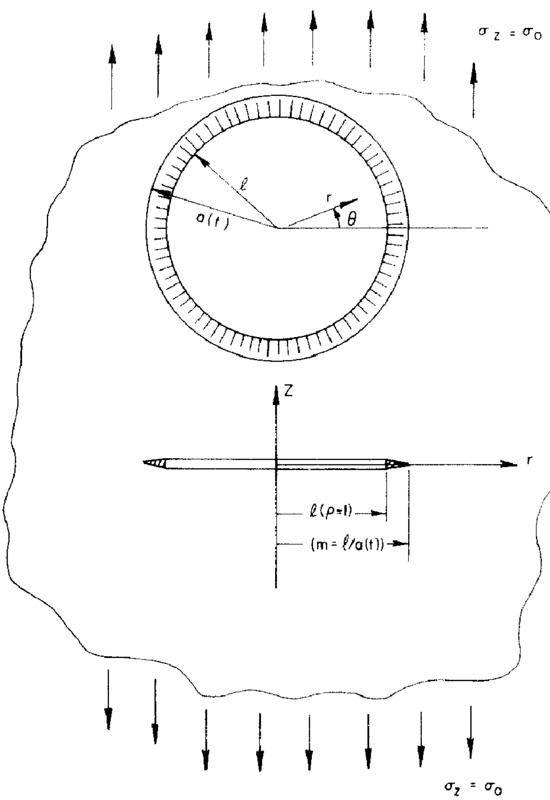

FIG. 1. Crack geometry.

can be written as

$$w(\rho, t) = w_o(\rho, t) + \int_0^t \frac{\dot{K}(\tau)}{K(o)} w_o(\rho, t-\tau)\,\mathrm{d}\tau. \tag{5}$$

Here, $w_o(\rho, t)$ is the associated short time elastic solution, $p(\rho, t)$ is the pressure applied at the crack surface $0 \leq r \leq a(t)$ (including the zones of plastic deformation), and $K(t)$ is defined as [15]

$$K(t) = \mathscr{L}^{-1}\left[\frac{2(2G_1^*(s)+G_2^*(s))}{s^2(G_1^*(s)+2G_2^*(s))G_1^*(s)}; s \to t\right] \tag{6}$$

stars denoting Laplace transformed quantities and $\mathscr{L}^{-1}$ denoting the inverse of the Laplace transform. Formulae of the same type are shown to be true for all components of displacement and strain tensors while the stresses are the same as in an elastic solid.

If one deals with viscoelastic behavior responses near the extremes of the spectrum and avoids the intermediate transition range, Poisson's ratio ν can be assumed nearly constant†. Then relation (6) simplifies to

$$K(t) = 2(1-\nu)D(t) \tag{7}$$

$D(t)$ being the creep compliance in shear.

† This restriction is not very severe; it allows dealing with hard polymers on the one hand and with soft rubbery ones on the other. In order to simplify analyses ν is often assumed constant over the full time range.

Before recording the expression for the stresses, it is appropriate to discuss the time dependence of the stress field from a physical viewpoint as it arises out of the time dependence of the material yielding at the crack tip. Under step loading there exists initially a domain of yield, the size of which is determined by the yield value $Y(0)$, and the stress distribution corresponds to that obtained for the elastic–plastic case [15]. The distribution of the σ_z stress in the vicinity of $l \leq r$ is indicated in Fig. 2(a). The ensuing creep increases the function χ and causes the value of the subsequent yield stress to drop† and consequently the size of the plastic zone to increase. This may be viewed as a discrete, incremental process giving rise to a stair-step like function of Fig. 2(b), and in the limit of many such increments as the continuous stress distribution in the same figure. Whether the actual stress distribution is like the one envisaged is not clear; nevertheless, the process described is consistent with the assumed model of time dependent plasticity.

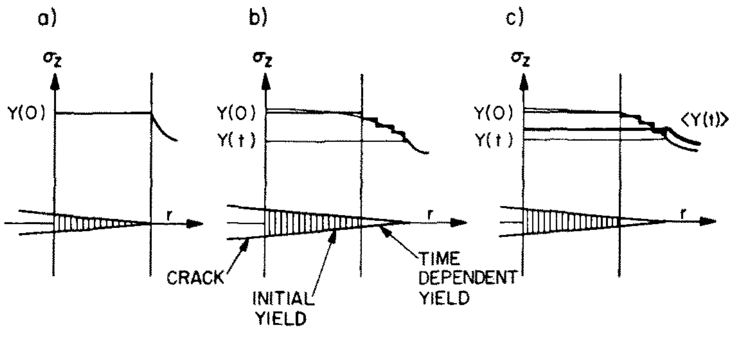

FIG. 2. Formation of the yielded zone and the distribution of stresses σ_z within this zone.

It turns out that the process just described leads to intractable mathematics and we shall therefore introduce a further simplification and represent the stress distribution in the yielded zone by a time dependent average $\langle Y(t)\rangle$ which is constant over the domain $l \leq r \leq a(t)$ as indicated in Fig. 2(c). Let this average be given by

$$\langle Y(t)\rangle = \tfrac{1}{2}\{Y(o)+Y(t)\} \tag{8}$$

where $Y(t)$ is evaluated for the strains at $r = a(t)$. With this physical clarification in mind we may now use the results obtained by Olesiak and Wnuk [15] and write down the stresses immediately. We shall do this for the case when the load is applied as a tensile stress at $z \to \infty$‡. Let $\rho = r/l$, $m(t) = l/a(t)$, $\kappa = \tfrac{1}{2}(1-2\nu)(1+\nu)$, $\lambda(t) = p(t)/\langle Y(t)\rangle$. We have then (cf. Ref. [18])

$$\left.\begin{aligned} \sigma_z &= 0 \\ \sigma_r &= p(t)(\kappa-1) \\ \sigma_\theta &= -p(t)(2\nu+\kappa) \end{aligned}\right\} \qquad 0 \leq \rho < m$$

† The associated unloading poses no difficulty in the formulation of the viscoelasticity problem.

‡ The case where the load is applied as a pressure at the crack surface is treated in detail in Ref. [18].

$$
\left.\begin{aligned}
\sigma_z &= \langle Y\rangle \\
\sigma_r &= \langle Y\rangle\left[(1-\lambda)(1-\kappa)+\kappa\left(\frac{m}{\rho}\right)^2\right] \\
\sigma_\theta &= \langle Y\rangle\left[(1-\lambda)(2\nu+\kappa)-\kappa\left(\frac{m}{\rho}\right)^2\right]
\end{aligned}\right\} \quad m \le \rho \le 1
$$

$$
\begin{aligned}
\sigma_z &= \frac{2\langle Y\rangle}{\pi}\left[\frac{\pi}{2}\lambda-\lambda\sin^{-1}\left(\frac{1}{\rho}\right)+\sin^{-1}\left(\frac{1-m^2}{\rho^2-m^2}\right)^{\frac{1}{2}}\right] \\
\sigma_r &= \frac{2\langle Y\rangle}{\pi}\left\{\left[1-\kappa+\kappa\left(\frac{m}{\rho}\right)^2\right]\sin^{-1}\left(\frac{1-m^2}{\rho^2-m^2}\right)^{\frac{1}{2}}-(1-\kappa)\lambda\sin^{-1}\left(\frac{1}{\rho}\right)\right\} \\
\sigma_\theta &= \frac{2\langle Y\rangle}{\pi}\left\{\left[2\nu+\kappa-\kappa\left(\frac{m}{\rho}\right)^2\right]\sin^{-1}\left(\frac{1-m^2}{\rho^2-m^2}\right)^{\frac{1}{2}}-(2\nu+\kappa)\lambda\sin^{-1}\left(\frac{1}{\rho}\right)\right\}.
\end{aligned} \quad \rho \ge 1 \tag{9}
$$

It can be observed that the stresses pass through a discontinuity at $\rho = m$. It should also be noted that the outer radius of the plastic zone is related to the (non-dimensional) load parameter [15] $\lambda(t) = p(t)/\langle Y(t)\rangle$ by

$$
m(t) = [1-\lambda^2(t)]^{\frac{1}{2}}. \tag{10}
$$

The yield value $\langle Y(t)\rangle$ is as yet unknown; in order to determine it we need to calculate the strains ε_{ij} at $r = a(t)$ from the stresses (9). Although the following calculations are possible without resorting to approximations, the restriction that the yield stress is much larger than the applied stress can simplify the analysis considerably. This simplification would be tantamount to ignoring the problem of general yield emanating from the crack tip and considering only limited yield prior to fracture. Then $\lambda(t) \ll 1$ and the stresses (9) at $r = a(t)$ reduce to

$$
\begin{aligned}
\sigma_z &= \langle Y(t)\rangle \\
\sigma_r &= \langle Y(t)\rangle \\
\sigma_\theta &= 2\nu\langle Y(t)\rangle
\end{aligned} \tag{11}
$$

while the corresponding short-time elastic strains ε_{ij}^e are

$$
\begin{aligned}
\varepsilon_z^e &= \frac{2\kappa}{E_g}\langle Y(t)\rangle \\
\varepsilon_r^e &= \frac{2\kappa}{E_g}\langle Y(t)\rangle \\
\varepsilon_\theta^e &= 0
\end{aligned} \tag{12}
$$

E_g being the glassy or short-time modulus.

The viscoelastic strains at the tip of the plastic zone are given by

$$
\begin{aligned}
\varepsilon_z^v = \varepsilon_r^v &= \frac{\langle Y(t)\rangle}{E_g}2\kappa+\frac{2\kappa}{E_g}\int_0^t\frac{\dot{K}(\tau)}{K(o)}\langle Y(t-\tau)\rangle\,\mathrm{d}\tau \\
\varepsilon_\theta^v &= 0
\end{aligned} \tag{13}
$$

and substitution of (12) and (13) into (3) renders the function χ after some manipulation as

$$\chi = \frac{2\sqrt{2}}{E_g}\varkappa \int_0^t \dot{\psi}(\tau)\langle Y(t-\tau)\rangle \, d\tau. \tag{14}$$

Here we have defined the normalized creep compliance $\psi(t) \equiv K(t)/K(o)$. Recalling that $2\langle Y(t)\rangle = Y(o)+Y(t)$ we can write now a non-linear integral equation for $Y(t)$ as

$$Y(t) = A+B\exp\left\{-\frac{\sqrt{(2)}\kappa C}{E}\left[(A+B)[\psi(t)-1]+\int_0^t \dot{\psi}(\tau)Y(t-\tau)\,d\tau\right]\right\}. \tag{15}$$

This expression can be reduced by two-fold differentiation to the non-linear differential equation

$$\frac{E_g}{\kappa\sqrt{(2)}C}\frac{\dot{Y}^2-(Y-A)\ddot{Y}}{(Y-A)^2} = 2(A+B)\ddot{\psi}(t)+\dot{\psi}(t)\dot{Y}. \tag{16}$$

With the definitions

$$y(t) = Y(t)-A, \qquad \alpha = 2\sqrt{(2)}\kappa\frac{C}{E_g}$$

$$P(t) = \frac{\alpha}{2}\dot{\psi}(t), \qquad Q(t) = \alpha(A+B)\ddot{\psi}(t)$$

this equation simplifies to

$$\dot{y}^2 - y\ddot{y} = y^2[P(t)\dot{y}+Q(t)]. \tag{17}$$

EFFECT OF TIME DEPENDENT YIELD

The solution of the non-linear differential equation (17) valid for $\lambda \ll 1$ poses a formidable task for general material properties $P(t)$ and $Q(t)$, and must be accomplished, in general, numerically. In one special, simple case however, the solution can be obtained analytically and in closed form, namely when the bulk material behaves as a Maxwell solid. In this case $D(t) = D(o)+t/\eta$, η being a constant (viscosity), one has from (7)

$$\begin{aligned} \psi(t) &= 1+\frac{t}{\eta D(o)} \\ \dot{\psi}(t) &= \frac{1}{\eta D(o)} \\ \ddot{\psi} &= 0. \end{aligned} \tag{18}$$

Equation (17) reduces then to

$$\ddot{Y}Y-\dot{Y}^2 = -\frac{\alpha}{2\tau_o}(Y-A)^2\dot{Y} \tag{19}$$

where $\tau_o = \eta D(o)$. Noting that

$$(\ddot{Y}Y - \dot{Y}^2/Y^2) = \frac{d}{dt}(\dot{Y}/Y),$$

equation (19) can be integrated directly, subject to the initial conditions of equation (15),

$$Y(0^+) = A+B$$

$$\dot{Y}(0^+) = -\alpha B(A+B)\psi(o) = -\alpha B\frac{Y(o)}{\tau_o}$$

to give the average yield stress

$$\langle Y(t)\rangle = \langle Y(\infty)\rangle\left\{1-\frac{B}{2(A+B)}\exp\left[-\frac{\alpha}{2\tau_o}(2A+B)t\right]\right\}^{-1}. \tag{20}$$

It may be verified by substitution of the definitions

$$\beta = \frac{B}{2A+B}$$

$$c = \sqrt{(2)}\kappa\frac{C(2A+B)}{E_g} = \frac{\alpha}{2}(2A+B)$$

$$\Phi(t) = 1+\beta-\beta\exp(-ct/\tau_o)$$

that (20) is reduced to

$$\langle Y(t)\rangle = Y(o)/\Phi(t). \tag{21}$$

Equation (21) gives the average stress in the plastic zone surrounding the penny-shaped crack. Figure 3 shows the decay of the stress in the plastic zone. For the material parameters we have chosen† $A = 100$ psi, $B = 25$ psi, $C = 400$, $\nu = 0{\cdot}3$, the three values of c, $1 \leq c \leq 10$ correspond to a range of Young's modulus of $5\times10^4 \leq E \leq 5\times10^5$ psi.

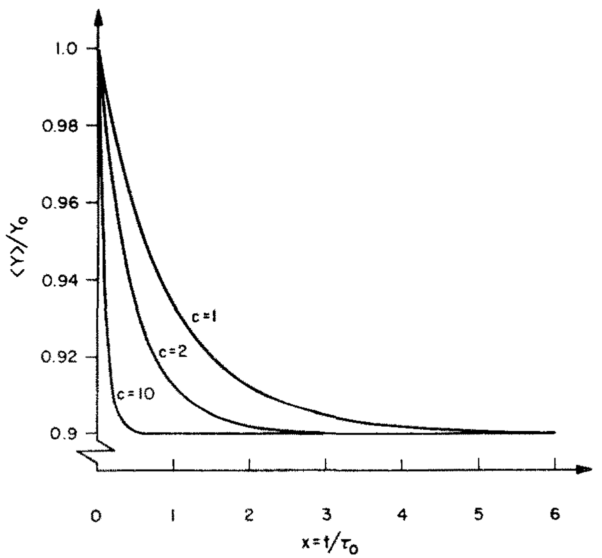

FIG. 3. Decrease of the yield stress prior to fracture.

† These values were taken from Ref. [19]. Although they have direct practical significance only for the filled polymer for which they were obtained, these values are physically not without meaning. In the absence of information on viscoplastic material properties, they are more significant than a mere guess.

Although the effect of C on the plastic relaxation is considerable, the same is not true when one considers the displacement growth at the tip of the crack. Following [18] it can be readily shown that the displacement $w(1, t)$ at the tip of the crack ($\rho = 1$) for step loading $p(t) = p_o H(t)$ is given by

$$w(1, t) \equiv w_o(t) = w(o)[\Phi(t) + \int_0^t \dot{\psi}(\tau)\Phi(t-\tau)\,\mathrm{d}\tau] \tag{22}$$

where

$$w(o) = \frac{2(1-\nu^2)lp_o^2}{\pi E_g Y(o)}.$$

Substitution of $\Phi(t)$, equation (21), renders for the Maxwell solid,

$$\frac{w_o(t)}{w(o)} = 1 + (1+\beta)\frac{t}{\tau_o} + \beta\left(1 - \frac{1}{c}\right)[1 - \mathrm{e}^{-ct/\tau_o}]. \tag{23}$$

This relation is illustrated in Fig. 4 and it is seen that the displacement is considerably less sensitive to variations in c than the yield stress.

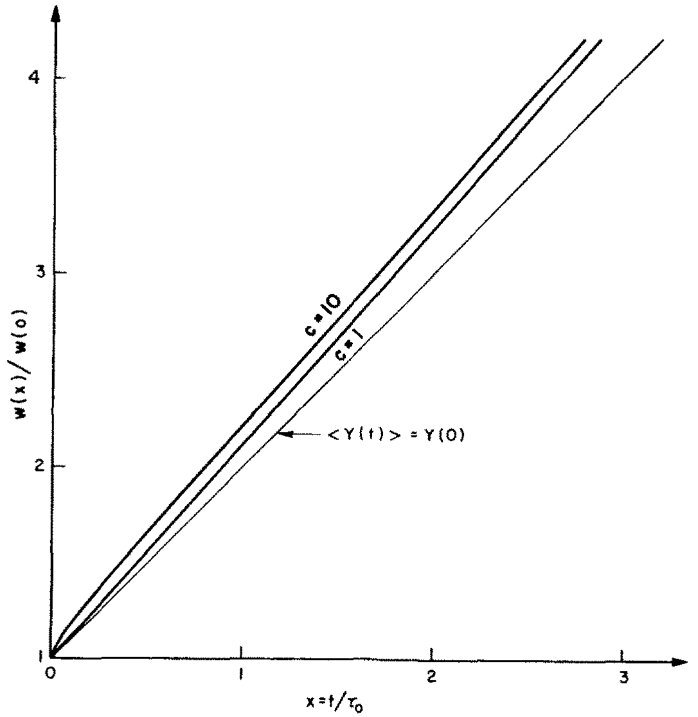

FIG. 4. Growth of displacement at the crack tip prior to fracture (solid line corresponds to time dependent yield stress, while fine line results when yield stress is assumed constant).

It should now be recalled that we adopted from the beginning a strain or displacement criterion of failure initiation. According to that criterion, crack propagation starts when the crack tip displacement $w_o(t)$ reaches the critical value w^* at time t^*, i.e. when

$$w_o(t^*) = w^*. \tag{24}$$

The time to failure is then obtained implicitly from (23) upon substituting (24)

$$\frac{w^*}{w(o)} = 1+(1+\beta)\frac{t^*}{\tau_o}+\beta\left(1-\frac{1}{c}\right)[1-\exp(-ct^*/\tau)]. \tag{25}$$

To relate $w^*/w(o)$ to the load p_o in a simple way let $\gamma = w^* Y_o$. It has been shown [20] that this is equal to the plasticity parameter in the Orowan–Irwin theory of fracture under limited, time-independent ductility. Furthermore, let

$$p_g^2 = \frac{\pi E_g Y(o)}{2(1-v^2)l} = \frac{\pi E_g \gamma}{2(1-v^2)l} \tag{26}$$

denote the Griffith stress p_g to cause fracture propagation upon load application in a brittle manner and without time delay. Upon using the definition of $w(o)$ following equation (22) and the definition (26), equation (25) may be rewritten as

$$\frac{p_o}{p_g} = \left\{1+(1+\beta)\frac{t^*}{\tau_o}+\beta\left(1-\frac{1}{c}\right)[1-\exp(-ct^*/\tau_o)]\right\}^{-\frac{1}{2}}. \tag{27}$$

This relation between the time to initiate fracture and the applied load is shown in Fig. 5 as Trace 1. Shown in the same figure is the result for constant, rather than time dependent, yield, Traces 2 and 3 corresponding to yield stresses at zero and infinite time respectively.

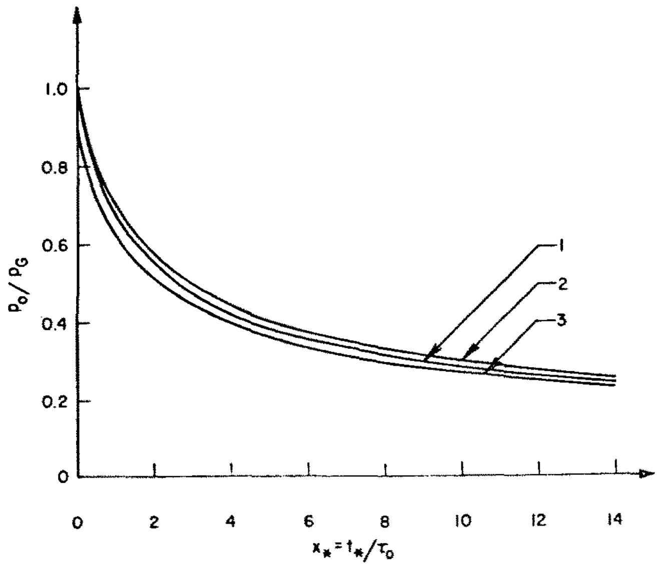

FIG. 5. Delayed fracture as a function of load level. 1 yield stress allowed to vary with time; 2 and 3 obtained under an assumption of the constant yield stress:

$\langle Y \rangle = Y_0$ curve 1 $\quad \langle Y \rangle = Y_\infty$ curve 3.

It is clear then that the decrease in yield stress with time accelerates the deformation at the crack tip and causes earlier failure than would be true if the initial yield stress were maintained. Thus a fracture prediction is conservative only if it is based on the constant, long time yield stress $Y(\infty)$ in which case one has

$$\frac{p_o}{p_g} = \left\{\frac{Y(\infty)}{Y(o)}\right\}^{\frac{1}{2}}\{1+t^*/\tau_o\}^{-\frac{1}{2}}. \tag{28}$$

We have now investigated the inception of fracture propagation in the presence of limited time dependent plasticity. Although use of more realistic material properties could lead to different numerical results the qualitative behavior should be the same. In spite of the restrictions imposed by the simple material representation, it appears that time dependent plasticity does not lead to gross deviations from what holds true for time-independent plastic behavior. With this qualitative feeling as an incentive we shall now consider the time-dependence of the fracture process in the presence of time-independent yield, but for more general viscoelastic behavior of the unyielded material than a simple Maxwell model.

DELAYED FRACTURE FOR TIME-INDEPENDENT YIELD

The simplification of time-independent yield properties eliminates the necessity of solving the non-linear differential equation (17) and allows therefore a more general representation for the bulk of the material. Furthermore, we need not necessarily restrict ourselves to low values of λ. The resulting expressions to determine the times of incipient crack propagation are so simple that their usefulness in applications may benefit from this simplicity more than they may suffer from their lack of a complete material representation.

The normal crack tip displacement $w(t)$ at $\rho = m$ is obtained from equation (5) after substituting the elastic solution [15]

$$w_o(m, t) = \frac{4(1-\nu^2)lY_o}{\pi E_g}\{1-(1-\lambda^2)^{\frac{1}{2}}\} \tag{29}$$

The result is, with $\lambda(t) = p(t)/Y_o$,

$$w(t) = w(m, t) = \frac{4(1-\nu^2)lY_o}{\pi E_g}\left\{1-[1-\lambda^2(t)]^{\frac{1}{2}}+\int_0^t \dot{\psi}(\tau)[1-[1-\lambda^2(t-\tau)]^{\frac{1}{2}}]\,\mathrm{d}\tau\right\}. \tag{30}$$

Let w^* be the value of $w(t)$ at the time of failure t^*; furthermore, define [18, 19]

$$\gamma \equiv w^* Y_o$$

$$p_g^2 = \frac{\pi E_g \gamma}{2(1-\nu^2)l}.$$

Then (30) may be written as

$$\left(\frac{p_g}{Y_o}\right)^2 = 2\left\{1-[1-\lambda^2(t)]^{\frac{1}{2}}+\int_0^{t^*} \dot{\psi}(\tau)\{1-[1-\lambda^2(t-\tau)]^{\frac{1}{2}}\}\,\mathrm{d}\tau\right\} \tag{31}$$

which relates the load history $\lambda(t)$ to the failure time t^*. Note that there exists a minimum crack size $\min[l] = l^*$ below which the applied load $p(t)$ would have to exceed the yield stress to cause failure. The size of l^* is given by the condition that $p_g \equiv Y_o$, so that

$$l^* = \frac{\pi E_g \gamma}{2(1-\nu^2)Y_o^2} = \frac{\pi E_g w^*}{2(1-\nu^2)Y_o}. \tag{32}$$

For cracks of initial length $l \leq l^*$ general yield will therefore occur rather than crack growth.

For a step load $p(t) = p_o H(t)$ equation (31) becomes, with $\lambda_o = p_o/Y_o$

$$\left(\frac{p_g}{Y_o}\right)^2 = 2\psi(t^*)\{1-[1-\lambda_o^2]^{\frac{1}{2}}\}. \tag{33}$$

If we define the inverse function of

$$\psi(t) \equiv D(t)/D(o) \quad \text{as} \quad t \equiv \psi^{-1}[D(t)/D(o)]$$

one may write the time of instability t^* explicitly from (33) as

$$t^* = \psi^{-1}\left[\frac{1}{2}\left(\frac{p_g}{Y_0}\right)^2 \frac{1}{1-(1-\lambda_0^2)^{\frac{1}{2}}}\right]. \tag{34}$$

This time t^* is a function of the crack size through $(p_g/Y_0)^2$ and of the applied load through $\lambda_0 = p_0/Y_0$ as long as $l > l^*$, no restrictions being placed on the size of the plastic zones at the crack periphery. The function ψ^{-1} is zero for arguments less than or equal to unity. Hence instantaneous fracture ensues if

$$\frac{1}{2}\left(\frac{p_g}{Y_0}\right)^2 \leq 1-(1-\lambda_0^2)^{\frac{1}{2}}. \tag{35}$$

On the other hand, if the reverse is true, i.e. if

$$\frac{1}{2}\left(\frac{p_g}{Y_0}\right)^2 > 1-(1-\lambda_0^2)^{\frac{1}{2}} \tag{36}$$

then t^* is greater than zero which means that some time will pass after load application before the crack starts to propagate. For illustrative purposes we show in Fig. 6 the time to

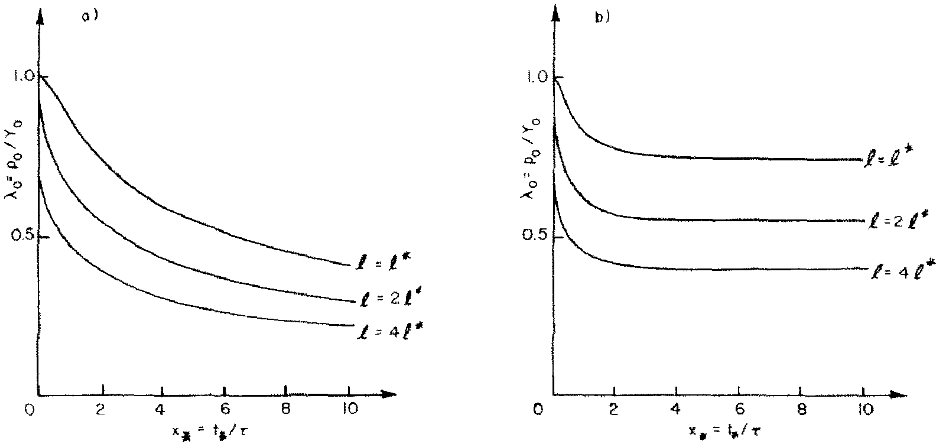

FIG. 6. Delayed fracture caused by a penny shaped crack. τ = relaxation time (a) Maxwell solid; (b) standard linear solid $E_0/E_\infty = \frac{3}{2}$.

failure for a Maxwell solid and a standard linear solid. The weakening effect of larger cracks is clearly illustrated.

If a crack is very much larger than the minimum size l^* fracture occurs at low load levels $\lambda_0 = p_0/Y_0 \ll 1$ and equation (31) may be written more simply as

$$\left(\frac{p_g}{Y_0}\right)^2 = \lambda^2(t^*) + \int_0^{t^*} \dot{\psi}(\tau)\lambda^2(t^*-\tau)\,\mathrm{d}\tau. \tag{37}$$

By multiplying both sides of this equation by Y_0^2 the yield stress vanishes from the equation. Therefore, the fracture resulting from large cracks at low load levels is nearly independent of the yield process at the tip of the crack. This result is well recognized for the rate-insensitive metals [20, 22]. For the particular case of step loading $p(t) = p_0 H(t)$ one obtains then the simple result

$$\left(\frac{p_g}{p_0}\right)^2 = \psi(t^*) \tag{38}$$

which equation yields immediate failure initiation ($t^* = 0$) if $p_0 = p_g$ and predicts infinite failure time when $(p_g/p_0)^2 = \psi(\infty)$. It follows that if $\psi(t)$ is bounded at infinity there exists a lower limit on p_0 below which no crack propagation occurs. This lower limit is

$$p_{\min} = p_g\{E_e/E_g\}^{\frac{1}{2}} \tag{39}$$

where E_e is the long-time equilibrium modulus. If $\psi(\infty)$ is not bounded, i.e. if $E_e = 0$, then such a limit does not exist and fracture may always occur after long times.

It should be noted, with a view toward applications of (38) that one need not know the value of p_g. Suppose one conducts tests on materials containing a crack (or several non-interacting cracks) of size l_1 and finds that a load p_{o1} produces failure in time t_1^*. Equation (38) can then be written as

$$\frac{\pi E_g \gamma}{2(1-\nu^2)l_1 p_{o1}^2} = \psi(t_1^*) \tag{40a}$$

and for any other load and crack size as

$$\frac{2E_g \gamma}{2(1-\nu^2)l p_o^2} = \psi(t^*). \tag{40b}$$

Division of (40a) by (40b) renders

$$l p_o^2 = \{l_1 p_{o1}^2 \psi(t_1^*)\}/\psi(t^*) \tag{41}$$

which equation would permit simple extrapolation of a minimum of experimental data to other loads and crack sizes.

Inasmuch as equations (37) and (38) do not contain the yield stress they may be also applied to materials which do not exhibit yield-like behavior provided the applied stresses are small compared to the intrinsic molecular strength of the material [20, 22]. Correspondingly, we show in Fig. 7 the prediction of failure initiation for Solithane 113 (50/50) the mechanical properties of which are well documented in Ref. [21]. It is interesting to note that a very similar result was obtained for the same material, by Williams [7] who considered an energy criterion of fracture for a spherical void under hydrostatic tension. In our

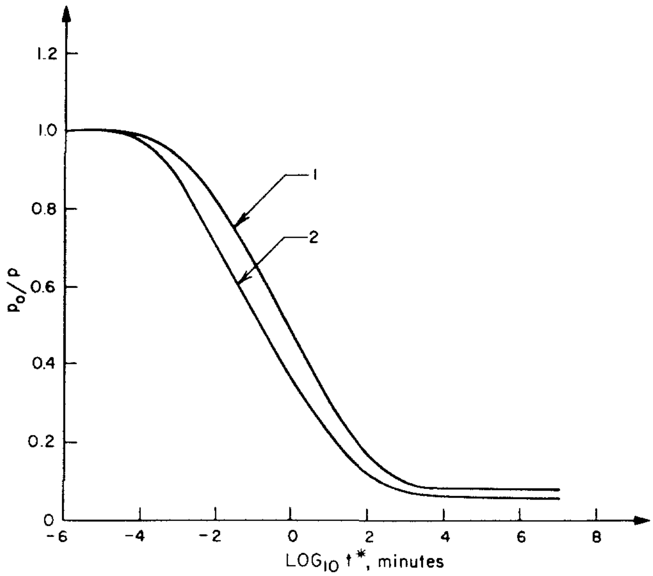

FIG. 7. Creep failure curves: referred to the glass transition temperature. 1—present paper equation (37). 2—Williams' result for a spherical void, Ref. [7].

current notation that result is

$$\left(\frac{p_g}{p_o}\right)^2 = 2\psi(t^*) - 1. \tag{42}$$

This equation is also represented in Fig. 7 for Solithane 113 (50/50).

In concluding this discussion of fracture initiation in viscoelastic materials from a penny-shaped crack we comment on the failure behavior in two-dimensional stress fields. It can be shown in a straightforward manner that for two-dimensional geometries the previous calculations follow through to give results which differ only in detail from those presented here. Indeed, equations (38–41) are identical. A more detailed comparison of the two and three-dimensional is presented in Ref. [18]. Similarly, the reader may refer to [18] for a discussion on the effect of a temperature and rate sensitive critical strain on displacement w^* at the tip of the crack.

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Абстракт—В некоторых твердых, вязкоупругих полимерах образование трещины связано с возникновением клина сетки волокнистых потресканий на конце трещины. Потресканный материал образуется с общей массы материала путем возникновения отверствий при заданных напряжениях которые можно сравнить с пределом текучести для металлов.

В работе обсуждается время, при котором дискообразная трещина, окруженная клинообразным кольцем потресканного метериала, начинается распространяться. Общая масса метериала рассматривается как вязкоупругая. Потресканный материал клина представляет вязкопластический модель Кроше. Оказывается, что зависящая от времени пластичность клина сокращает время разрушения, по сравнению с теорией пластичности, не зависящей от времени.